ON SUBHARMONIC OSCILLATIONS OF A PENDULUM

PMM Vol. 35, №2, 1971, pp. 343-348

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(Received March 23, 1970)

Subharmonic oscillations of a pendulum excited by horizontal oscillations of its suspension in the case of simple harmonic excitation are investigated.

The motion of a mathematical pendulum excited by oscillations of its point of suspension has been studied by many authors [1−6]. The influence of the vertical oscillations of the point of suspension is ordinarily considered but there are papers devoted to the influence of the horizontal oscillations [7] and of oscillations of more general form [8].

A certain supplement of Struble [9, 10] to existing asymptotic methods in the theory of nonlinear oscillations [11] is used herein.

1. The motion of a mathematical pendulum excited by the horizontal oscillations of its suspension is defined by the equation

$$\varphi^{\cdot\cdot} + \frac{g}{l}\sin\varphi = -\frac{1}{l}x^{\cdot\cdot}\cos\varphi \qquad (1.1)$$

where the dots denote differentiation with respect to time, l is the length of the pendulum, φ the angle of deflection from the vertical, and x the displacement of the pendulum suspension.

Let us examine relatively small deflections of the pendulum. Let us assume

$$\varphi = \varepsilon^{1/2} z, \quad \frac{x^{\cdot\cdot}}{l} = -\varepsilon^{1/2} a_1 p^2 \cos pt, \quad p\,(l/g)^{1/2} = \beta$$

$$a = a_1\beta^2 \quad (\varepsilon \text{ is a small parameter}) \qquad (1.2)$$

If the dimensionless time $\tau = (g/l)^{1/2} t$ is introduced into (1.1), we obtain the equation

$$z'' + z = a\cos\beta\tau + \varepsilon\,({}^1/_6\, z^3 - {}^1/_2\, a z^2 \cos\beta\tau) \qquad (1.3)$$

The primes here denote differentiation with respect to τ and terms containing higher powers of ε than the first are discarded. Let us seek the solution of (1.3) in the form

$$z = A\cos(\tau - \psi) + \frac{a}{1-\beta^2}\cos\beta\tau + \varepsilon z_1 + \varepsilon^2 z_2 + \ldots \qquad (1.4)$$

Here A, ψ are slowly varying functions of τ and $z_1, z_2, \ldots$ are additive corrections expressed uniquely in terms of A, ψ and τ. For the sake of brevity, the case is considered when the excitation frequency p is approximately thrice the natural frequency $(g/l)^{1/2}$, i.e. $\beta \approx 3$. Substituting (1.4) into (1.1) we find

$$(A'' + 2A\psi' - A\psi'^2)\cos(\tau - \psi) + (A\psi'' - 2A' + 2A'\psi')\sin(\tau - \psi) + \varepsilon(z_1'' + z_1) +$$
$$+ \varepsilon^2 (z_2'' + z_2) + \ldots = \qquad (1.5)$$
$$= \varepsilon\{[{}^1/_8 A^3 + {}^1/_4 a^2 (2\beta^2 - 1)(1 - \beta^2)^{-2} A]\cos(\tau - \psi) + {}^1/_8\, \alpha\beta^2 (1 - \beta^2)^{-1} A^2 \cos \times$$
$$\times[(2 - \beta)\tau - 2\psi] + {}^1/_{24} A^3 \cos(3\tau - 3\psi) + [{}^1/_4\, a\beta^2 (1 - \beta^2)^{-1} A^2 + {}^1/_8\, a^3 (3\beta^2 - 2) \times$$
$$\times (1 - \beta^2)^{-3}]\cos\beta\tau + {}^1/_{24}\, a^3 (3\beta^2 - 2)(1 - \beta^2)^{-3}\cos 3\beta\tau + {}^1/_8\, a\beta^2 (1 - \beta^2)^{-1} A^2 \cos \times$$
$$\times[(2 + \beta)\tau - 2\psi] + {}^1/_8\, a^2 (2\beta^2 - 1)(1 + \beta^2)^{-2}\cos[(2\beta - 1)\tau + \psi] + {}^1/_8\, a^2 (2\beta^2 - 1) \times$$
$$\times (1 + \beta^2)^{-2}\cos[(2\beta + 1)\tau - \psi]\}$$

Here terms containing higher powers of ε than the first have been discarded. Utilizing the identity

$$\cos[(2-\beta)\tau - 2\psi] = \cos\lambda\cos(\tau-\psi) + \sin\lambda\sin(\tau-\psi),\quad \lambda = (3-\beta)\tau - 3\psi \quad (1.6)$$

and comparing corresponding terms in (1.5), we obtain

$$A'' + 2A\psi' - A\psi'^2 = \varepsilon[{}^1/_8 A^3 + {}^1/_4 a^2(2\beta^2-1)(1-\beta^2)^{-2}A] + {}^1/_8\,\varepsilon a\beta^2(1-\beta^2)^{-1}A^2\cos\lambda$$
$$2A' + A\psi'' + 2A'\psi' = {}^1/_8\,\varepsilon\alpha\beta^2(1-\beta^2)^{-1}A^2\sin\lambda \quad (1.7)$$

and

$$z_1'' + z_1 = [{}^1/_4\,a\beta^2(1-\beta^2)^{-1}A^2 + {}^1/_8\,a^3(3\beta^2-2)(1-\beta^2)^{-3}]\cos\beta\tau + {}^1/_{24}A^3\cos(3\tau - 3\psi) + {}^1/_{24}\,a^3(3\beta^2-2)(1-\beta^2)^{-3}\cos3\beta\tau + {}^1/_8\,\alpha\beta^2(1-\beta^2)^{-1}A^2\cos[(2+\beta)\tau - 2\psi] + {}^1/_8\,a^2(2\beta^2-1)(1-\beta^2)^{-2}\cos[(2\beta-1)\tau+\psi] + {}^1/_8\,a^2(2\beta^2-1)(1+\beta^2)^{-2}\times\cos[(2\beta+1)\tau-\psi] \quad (1.8)$$

Equations (1.7) are variational, and (1.8) the perturbation equation.

If the "detuning" $|3-\beta|$ is a small first order quantity in ε, then any solution of the system

$$dA/d\tau = -{}^1/_8\,\varepsilon a\beta^2(1-\beta^2)^{-1}(\beta-1)^{-1}A^2\sin\lambda \quad (1.9)$$
$$d\psi/d\tau = {}^1/_{16}\,\varepsilon A^2 + {}^1/_8\,\varepsilon a^2(2\beta^2-1)(1-\beta^2)^{-2} + {}^1/_8\,\varepsilon a\beta^2(1-\beta^2)^{-1}(\beta-1)^{-1}A\cos\lambda$$

will satisfy the system (1.7) to the accuracy of first order terms in ε. The domain of subharmonic resonance is thereby defined, and it is convenient to write the system (1.9) as equations of the autonomous system

$$dA/du = A^2\sin\lambda,\qquad d\lambda/du = 2n - 4\alpha A^2 + 3A\cos\lambda \quad (1.10)$$

Here

$$u = 8(\beta^2-1)(\beta-1)a^{-1}\beta^{-2}\varepsilon\tau,\qquad \alpha = {}^3/_8(\beta^2-1)(\beta-1)a^{-1}\beta^{-2}$$
$$n = {}^1/_2[(3-\beta)\varepsilon^{-1} - {}^3/_8 a^2(2\beta^2-1)(1-\beta^2)^{-2}]\,8(\beta^2-1)(\beta-1)a^{-1}\beta^{-2}$$

The changes in A and λ will evidently be slow since u is proportional to the slow time $\varepsilon\tau$. Eliminating u from (1.10) we obtain

$$(2n - 4\alpha A^2 + 3A)\cos\lambda\,dA - A^2\sin\lambda\,d\lambda = 0 \quad (1.11)$$

This equation has the general integral

$$nA^2 - \alpha A^4 + A^3\cos\lambda = c \quad (1.12)$$

where c is the constant of integration.

2. Let us investigate the phase trajectories for the autonomous system (1.10) in the xy-plane for which $x = A\cos\lambda$, $y = A\sin\lambda$, i.e. A and λ will be natural polar coordinates. The phase trajectories are defined by (1.12), and they are all symmetrical relative to the x-axis. Let us first determine the singularities of the system (1.10). From the conditions

$$dA/du = 0 \qquad d\lambda/du = 0 \quad (2.1)$$

we find

$$\sin\lambda = 0,\quad 2n - 4\alpha A^2 \pm 3A = 0 \quad (2.2)$$

It is now seen that the singularities are on the x-axis and determined as the roots of the quadratic equation (2.2). Moreover, the origin $A = 0$ is also a singular point. We obtain a graphical picture of the location of the singularities by representing (2.2) as

$$2n = f(x) = 4\alpha x^2 - 3x \quad (2.3)$$

and solving it graphically (Fig. 1).

For $n < -{}^9/_{32}\alpha^{-1}$ the equation has no real roots. For $n = -{}^9/_{32}\alpha^{-1}$ there appears one double root $x = {}^3/_8\alpha^{-1}$. For $-{}^9/_{32}\alpha^{-1} < n < 0$ there are two positive roots. For

$n = 0$ the lesser root becomes zero, and for $n > 0$ there are one negative and one positive root (the negative root is the singularity on the negative x-half-axis).

Depending, therefore, on the magnitude of n, which is expressed in terms of the detuning $|3 - \beta|$, the following fundamental cases can be established.

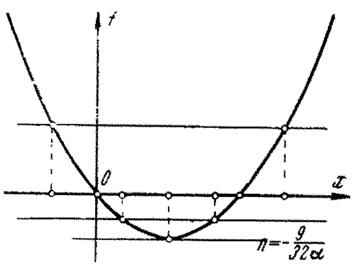

Fig. 1

1°. For $n < -{}^9/_{32}\alpha^{-1}$. In this case there exists just one singularity, the origin which is a center (Fig. 2a). In the boundary case $n = -{}^9/_{32}\alpha$ there are two singularities, the origin which is again a center, and $A = {}^3/_8 2$, $\lambda = 0$, which turns out to be an extraordinary singularity, whose index is zero (Fig. 2b).

2°. For $-{}^9/_{32}\alpha^{-1} < n < 0$. In this case there are three singularities, the origin (a center), a point corresponding to the least root of (2.3) (a saddle point), and a point corresponding to the larger root of (2.3) (a center). A separatrix in the form of a "figure-eight" enclosing the two centers passes through the saddle point. The phase trajectories for this case are shown in Fig. 2c. In the limit case of $n = 0$ the saddle point and the center corresponding to the origin merge to form an unstable critical point analogous to point $A = {}^3/_8\alpha$, $\lambda = 0$ in the limiting case $n = -{}^9/_{32}\alpha^{-1}$. The phase trajectories for this case are shown in Fig. 2d.

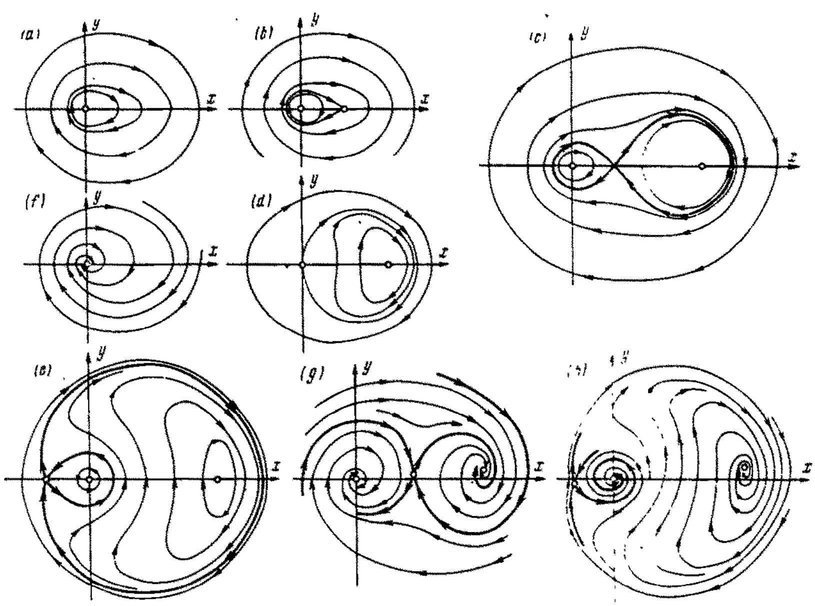

Fig. 2

3°. For $0 < n$. Here, as in case 2°, there are three singularities. The origin and a point corresponding to the positive root are centers, while the point corresponding to the negative root is a saddle point. Three families of phase trajectories are separated by the separatrix starting from the saddle point and reentering it (Fig. 2e).

The phase trajectories transform into circles as $c \to \infty$ for all cases. The centers and saddle points on the saddle point determined by (2.3) are bifurcation amplitudes corresponding to stable and unstable subharmonic solutions of (1.3). The closed phase trajectories describe long-period oscillations of amplitude A (beats) while the separatrices correspond to transition values of A.

If a slight damping is inserted into the system, then the centers on the phase plane transform into asymptotically stable foci, while the saddle points do not alter substantially. The subharmonic solutions corresponding to the foci will be asymptotically stable. The phase trajectories now describe a very slow transition mode of the motion. Cases 1°, 2°, 3° in the presence of damping are shown in Fig. 2f, g, h, respectively. It must be noted that the focus (and center) $A = 0$ corresponds to the harmonic solution (1.4) which has no first term, rather than to the subharmonic solution.

3. Let us find the amplitude A as a function of u. From (1.10) and (1.12) we obtain

$$\frac{d\mu}{\pm\sqrt{\mu^3 - (c - n\mu + \alpha\mu^2)2}} = 2du \qquad (\mu = A^2) \tag{3.1}$$

Let us consider the polynomial

$$G(\mu) = \mu^3 - (c - n\mu + \alpha\mu^2)^2 \tag{3.2}$$

The roots of the polynomial (3.2) agree with the positive roots (for A^2) of (1.12) with $\cos\lambda = \pm 1$. For different values of c and n the polynomial (3.2) has four real roots or two real and two complex roots, i.e. it can be written in the form

$$G(\mu) = -\alpha^2(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)(\mu - \mu_4), \quad \mu_1 > \mu_2 > \mu_3 > \mu_4 > 0 \tag{3.3}$$

or

$$G(\mu) = -\alpha^2(\mu - \mu_1)(\mu - \mu_2)[(\mu - \nu)^2 + \omega^2], \quad \mu_1 > \mu_2 > 0$$
$$\nu = {}^1/_2[\alpha^{-2} + 2n\alpha^{-1}\alpha - (\mu_1 + \mu_2)], \quad \omega^2 = c^2\alpha^{-2}\mu_1\mu_2 - \nu^2 \tag{3.4}$$

The polynomial (3.2) will have the form (3.3) if two phase trajectories intersecting the x-axis exist for some value of c. The first at points with the polar radii $A_3 = \sqrt{\mu_3}$, $A_4 = \sqrt{\mu_4}$, and the second at points with the polar radii $A_1 = \sqrt{\mu_1}$, $A_2 = \sqrt{\mu_2}$. The polynomial (3.2) will have the form (3.4) if for some value of c there exists just one phase trajectory intersecting the x-axis at points with the polar radii $A_1 = \sqrt{\mu_1}$, $A_2 = \sqrt{\mu_2}$.

The real roots are found directly in constructing the appropriate phase trajectory. It is easy to show that the function $G(\mu)$ has the form (3.4) for all phase trajectories in case 1°, as well as for the majority of phase trajectories in cases 2° and 3°. It can be shown that the polynomial $G(\mu)$ has the form (3.3) in case 3° for all phase trajectories which close around the origin, as well as for their corresponding phase trajectories (for the same values of c) which close around the other center. In case 2° the polynomial $G(\mu)$ has the form (3.3) for all phase trajectories which close around the other center as well as for their corresponding phase trajectories which close around the origin for $-{}^9/_{32}\alpha^{-1} < n < -{}^1/_4\alpha^{-1}$. The polynomial $G(\mu)$ will have the form (3.3) also for $-{}^1/_4\alpha^{-1} < n < 0$ for all phase trajectories which close around the origin and for corresponding phase trajectories which close around the other center.

Let us first examine the case when $G(\mu)$ has the form (3.3). Let us set

$$k^2 = \frac{(\mu_3 - \mu_4)(\mu_2 - \mu_1)}{(\mu_3 - \mu_1)(\mu_2 - \mu_4)}, \qquad l^2 = \frac{4}{(\mu_1 - \mu_3)(\mu_2 - \mu_4)} \tag{3.5}$$

Then utilizing [12], we obtain from (3.1) for μ in the range $\mu_4 \leqslant \mu \leqslant \mu_3$

$$\mu = \frac{\mu_4(\mu_1-\mu_3)+\mu_1(\mu_3-\mu_4)\,\mathrm{sn}^2 U}{\mu_1-\mu_3+(\mu_3-\mu_4)\,\mathrm{sn}^2 U} \qquad \left(U=\frac{2\alpha}{l}(u-u_0)\right) \tag{3.6}$$

Here the modulus k of the Jacobi elliptic function sn is defined by (3.5) and u_0 is the value of the parameter u for $\mu = \mu_4$ For μ in the range $\mu_2 \leqslant \mu \leqslant \mu_1$ we have

$$\mu = \frac{\mu_2(\mu_1-\mu_3)-\mu_3(\mu_1-\mu_2)\,\mathrm{sn}^2 U}{\mu_1-\mu_3-(\mu_1-\mu_2)\,\mathrm{sn}^2 U} \qquad \left(U=\frac{2\alpha}{l}(u-u_0)\right) \tag{3.7}$$

Here the modulus k has the same value as in (3.6), and u_0 is the value of u for $\mu = \mu_2$.

The period of the long-period oscillations of amplitude A with respect to the time τ is defined by the formula

$$\varepsilon T = \frac{la\beta^2}{8(\beta^2-1)(\beta-1)}K(k) \tag{3.8}$$

Here $K(k)$ is the complete elliptic integral of the first kind in Legendre form, of modulus k. It is seen that the period of variation of A for the two cases (3.6) and (3.7) is the same although the motions themselves are completely distinct.

Let us now examine the case when $G(\mu)$ has the form (3.4). Here, following [12] we use the notation

$$\operatorname{tg} p = \frac{\mu_1-\nu}{\omega}, \quad \operatorname{tg} q = \frac{\mu_2-\nu}{\omega}, \quad k^2=\sin^2\frac{p-q}{2}, \quad l = -\frac{(\cos p\cos q)^{1/2}}{\omega} \tag{3.9}$$

For μ in the range $\mu_2 \leqslant \mu \leqslant \mu_1$ we obtain

$$\mu = \frac{\mu_1\cos p+\mu_2\cos q+(\mu_1\cos p-\mu_2\cos q)\,\mathrm{cn}\,U}{\cos p+\cos q+(\cos p-\cos q)\,\mathrm{cn}\,U} \qquad \left(U=\frac{2\alpha}{l}(u-u_0)\right) \tag{3.10}$$

Here the modulus k of the Jacobi elliptic function cn and the quantity l are defined by (3.9), while u_0 is the value of the parameter u for $\mu = \mu_1$.

The period of oscillations of amplitude A is defined by the formula

$$\varepsilon T = -\frac{la\beta^2}{4(\beta^2-1)(\beta-1)}K(k) \tag{3.11}$$

where the modulus of the complete elliptic integral and l are defined by (3.9). After determining A as a function of u and τ, by using (1.9) we can determine ψ as a function of u and τ.

The solution of the first approximation equations (1.7) and (1.8) is completed by determining the additive perturbation z_1. From (1.8) we obtain

$$\begin{aligned} z_1 = {}& {}^1/_8 a[2\beta^2(1-\beta^2)^2A^2+(3\beta^2-2)a^2](1-\beta^2)^{-4}\cos\beta\tau - {}^1/_{192}A^3\cos(3\tau-3\psi) + \\ &+ {}^1/_{24}(3\beta^2-2)a^3(1-\beta^2)^{-3}(1-9\beta^2)^{-1}\cos 3\beta\tau - {}^1/_8 a\beta^2A^2(1-\beta^2)^{-1}(\beta+1)^{-1}\cos\times \\ &\times[(2+\beta)\tau-2\psi] + {}^1/_{32}a^2(2\beta^2-1)(1+\beta^2)^{-2}(1-\beta)^{-1}\beta^{-1}\cos[(2\beta-1)\tau+\psi] - \\ &- {}^1/_{32}a^2(2\beta^2-1)(1+\beta^2)^{-2}(\beta+1)^{-1}\beta^{-1}\cos[(2\beta+1)\tau-\psi] \end{aligned} \tag{3.12}$$

In any case z_1 is the higher harmonics of the motion. They are not essential for the general representation of the character of the motion. The main term $A\cos(\tau-\psi)$ reflects the nature of the motion.

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Translated by M. D. F.

ON SYNTHESIS OF STABILITY OF SYSTEMS BY THE METHOD OF NONLINEAR PROGRAMING

PMM Vol. 35, №2, 1971, pp. 348-351

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(Received April 1, 1970)

A method of successive computation of the parameters governing the equation for stabilization of linear systems based on the ideas of nonlinear programing and reducing to minimization of the original functional is described. We do not succeed in presenting a rigorous mathematical foundation.

1. Let the perturbed motion of a stationary linear control system be described by the set of differential equations

$$dX/dt = AX + BU \tag{1.1}$$

Here X is the column vector of the fundamental variables; A is a square ($n \times n$) matrix, B is the column vector of the control efficiency coefficients, and U is a scalar of the controlling effect of the regulator.

It is assumed that the system (1.1) satisfies the controllability conditions. The matrices A, B are not degenerate, and the matrix $\varphi = \| B, AB, A^2B, \ldots, A^{n-1}B \|$ is of rank n and consists of n linearly independent vectors. It is required to seek the control law

$$U = CX \tag{1.2}$$

assuring asymptotic stability of the unperturbed motion $X = 0$. It is assumed that the matrix C has the form of a row vector and yields a square ($n \times n$) matrix in the product BC . Substituting (1.2) into the system (1.1) we obtain